Lecture 20: More on Gravitational Fields and Potential

Graphical Representation of Force and Potential

- We can gain an intuitive picture of a force field and potential as follows:
  1. At a random point in space, measure the force vector, which can be represented by an arrow
  2. Move a small distance in the direction of the force, and repeat the measurement.
  3. Repeat steps 1 and 2 many times
  4. Draw a line connecting all the arrows
- The line you've drawn is called a "line of force"
- By repeating the while process starting from different points, one creates a picture of the force
- Note that lines of force can never cross one another

- Once the lines of force are drawn, trace out a path that is perpendicular to all lines of force
- Recall that the work done by the force is equal to the change in potential energy:

$$W = \Delta U$$

- But on the path we've drawn, the direction of motion is always perpendicular to the force, so:

$$W = \int \mathbf{F} \cdot d\mathbf{r} = 0$$

- This means that $\Delta U = 0$, so that U is constant everywhere on the path
- Such a path is called an *equipotential surface*

Escape Velocity

- Using the concept of gravitational potential, we can determine how much initial velocity an object must have if it is to escape the gravitational field of another object
  - "Escape" means it's able to move infinitely far from the object generating the field
- For example, the moon is "infinitely" far from the Earth, so the Apollo spacecraft needed to escape from the Earth's gravitational field to get there
- Initial state: near the surface of the Earth, with some velocity v
- Final state: infinitely far from the Earth, with (at least) zero velocity

$$E_i = \frac{1}{2}mv^2 - \frac{GMm}{R_E} = E_f \geq 0 - \frac{GMm}{\infty} = 0$$

$$v \geq \sqrt{\frac{2GM}{R_E}}$$

Gauss' Law For Gravitation

- We can define the gravitational field flux through a surface as:

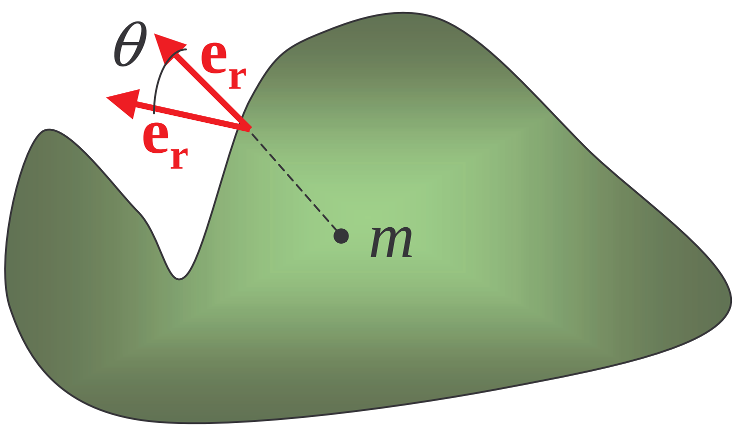
$$\Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} da$$

where **n** is a unit vector perpendicular to the surface at each point

- For the case of a closed surface around a point mass *m*, we have:

$$\Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} da = \int_S \left(\frac{-Gm}{r^2} \right) \mathbf{e_r} \cdot \mathbf{n} da$$

$$= -Gm \int_S \frac{\cos\theta}{r^2} da$$



- If we take the special case where the surface is a sphere, the integral is easy:

$$\Phi_m = -Gm\int_S \frac{1}{r^2}\,da = \frac{-Gm}{r^2}\int_S da$$
$$= -4\pi Gm$$

- Physically, we can interpret the flux as the "number" of gravitational field lines passing through the surface
  - But since the lines start at a point, and extend an infinite distance away, the flux can't depend on the shape of the surface that encloses our point
- Thus we have the *general* result, for any closed surface around a point mass,

$$\boxed{\Phi_m = -4\pi Gm}$$

Note that it also doesn't matter *where* the mass is

- This can be easily extended to the case where there are *N* point masses inside the surface. Since the field is linear,

$$\Phi_m = -4\pi G \sum_{i=1}^{N} m_i$$

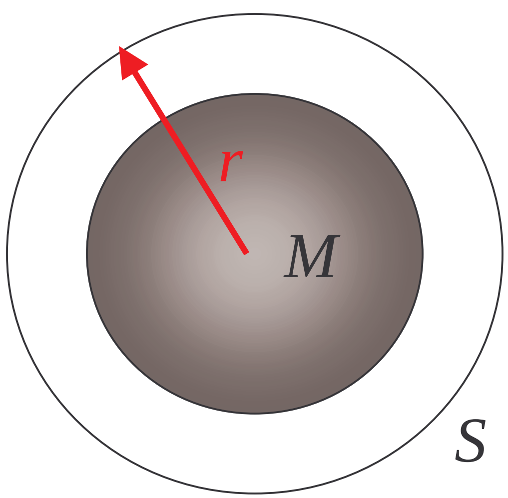
- For a continuous distribution of matter, it's:

$$\Phi_m = -4\pi G \int_V \rho(\mathbf{r})\, dv$$

- For problems involving symmetric distributions of matter, Gauss' Law is a useful shortcut to finding the field

Example: Field Due to a Sphere

- Assume we have a spherically symmetric mass distribution (with the density varying as a function of the distance from the sphere's center)
- We want to find the field at any point external to the sphere
  - The symmetry of the problem makes is a good candidate for Gauss' Law
  - We should choose a surface that reflects this symmetry:



$$\int_S \mathbf{g} \cdot \mathbf{n} da = -4\pi G M$$

From symmetry, we know that:

$\mathbf{g} = g_r(r)\mathbf{e_r}$ and $\mathbf{n} = \mathbf{e_r}$

- Therefore, the integral becomes much simpler:

$$\int_S \mathbf{g} \cdot \mathbf{n} da = \int_S g(r) da = 4\pi r^2 g(r)$$

$$4\pi r^2 g(r) = -4\pi GM$$

$$g(r) = \frac{-GM}{r^2}$$

- From this, we see that the field due to a sphere is exactly the same as if all the mass were concentrated at the center of the sphere
  - This is not true for other shapes
- This result was very important for Newton, since it justified his treatment of the Earth as a point mass when calculating the motion of the moon

Poisson's Equation

- We can look at Gauss' Law another way to find another important property of the gravitational potential:

$$\int_S \mathbf{g} \cdot \mathbf{n} da = \int_V \nabla \cdot \mathbf{g} dv$$

This is just the divergence theorem

$$= \int_V \left(\nabla \cdot \nabla \Phi \right) dv = \int_V \nabla^2 \Phi dv$$

$$= -4\pi G \int_V \rho\left(\mathbf{r}'\right) dv$$

- For the last relation to hold for an arbitrary volume V, the integrands must be the same everywhere:

$$\nabla^2 \Phi = -4\pi G M \rho\left(\mathbf{r}'\right)$$

Laplace's Equation

- In the special case where there is no material in a region of space, the potential in that region satisfies Laplace's Equation:

$$\nabla^2\Phi = 0$$

- Intuitively, this is nothing more than the statement that field lines can't start (or end) in a region where there is no mass
- Mathematically, this gives us a way to determine the potential in any mass-free region
  - As long as the bounday conditions (the value of the potential at the edges of the region) are specified
- In practice, this equation is more useful in calculating *electric* potentials than for gravitational potentials…