The Special Theory of Relativity

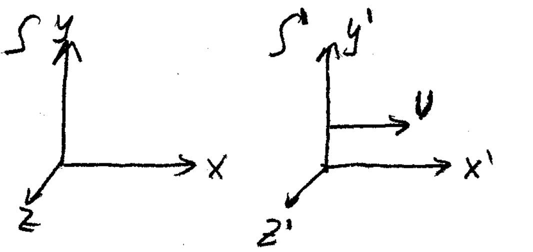
The postulates:

1. The laws of physics have the same form in all inertial reference frames

   or: It is impossible by performing experiments in a given inertial system to decide if this inertial system is "at rest" or "moving with constant velocity.

2. The speed of light in vacuum is c in all directions in all inertial reference frames, and is independent of the state of motion of the source and the observer.

The Galilei Transformation



$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

The inertial system S' is moving with velocity v with respect to the inertial system S, v constant.

Transformation of velocities:

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \Leftrightarrow V_x' = V_x - v \quad \text{and} \quad V_x = V_x' + v$$

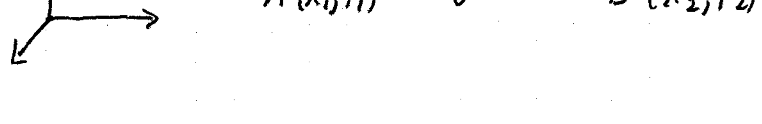
The Maxwell equations of electromagnetism are not invariant by a Galilei transformation. One (or both) of these two theories must be altered.

Einstein's choice: To alter the Galilei transformation, as the 2. postulate is in agreement with Maxwell.

As Newton's laws (especially N2) are invariant by the Galilei transformation, they also must be altered.

The Galilei transformation assumes <u>absolute simultaneity</u> ($t' = t$), but defining this concept on realistic physical considerations is not possible.

Realistic definition of simultaneity ~~must~~ could be based on light signals, see Giancoli.



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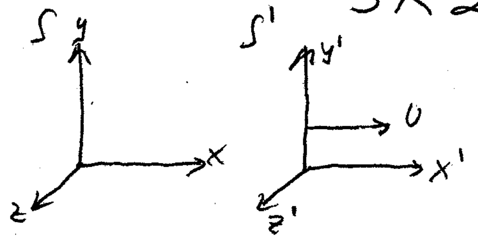
Simple derivation of the Lorentz Transformation.



We define S' such that the origins of S and S' pass each other at $t = t' = 0$.

At $t = t' = 0$ in (0,0) of S and S' an explosion occurs. The light signal from this explosion propagates out in S and S'.

Einsteins postulates:

1. The observers of S and S' must describe the propagation of the wavefront using equations of the same form.

2. The velocity of the wavefront must be c in S and S'.

The equations of the wavefront (spheres of radius ct)

$$S: \quad x^2 + y^2 + z^2 - c^2t^2 = 0$$

$$S': \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

Simplifying assumptions: (*)

$$x' = \gamma(u)(x - ut) \qquad \gamma(u) \approx 1 \text{ if } u \ll c$$

$$t' = A(u)\,t + B(u)\,x \qquad A \approx 1 \text{ and } B \approx 0 \text{ if } u \ll c.$$

To find the functions $\gamma(u)$ and $A(u)$ and $B(u)$ we must substitute into the "wave-equations":

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

or $$x^2+y^2+z^2-c^2t^2 = \gamma^2(x-ut)^2 + y^2 + z^2 - c^2(At+Bx)^2$$

or $$x^2+y^2+z^2-c^2t^2 = (\gamma^2 - B^2c^2)x^2 + y^2 + z^2 + (\gamma^2u^2 - A^2c^2)t^2 - 2(\gamma^2 u + ABc^2)xt$$

This leads to 3 equations:

$$\left.\begin{aligned} \gamma^2 - B^2c^2 &= 1 \\ A^2c^2 - \gamma^2u^2 &= c^2 \\ \gamma^2 u + ABc^2 &= 0 \end{aligned}\right\}$$

Solving:

$$\gamma = A = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$B = -\frac{u\gamma}{c}$$

(*) No transforms of the "vertical" coordinates y and z – marking vertical sticks would else violate the 1. postulate ---

The Lorentz Transformations

(Skipping $y' = y$, $z' = z$)

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \gamma(x - vt) \qquad \left(\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} = \gamma\left(t - \frac{v}{c^2}x\right)$$

and (solving for t, x or regarding S as moving with velocity $-v$ with respect to S':

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \gamma(x' + vt')$$

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}} = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Transformation of velocities.

Observe that: $\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) = \gamma\left(1 - \frac{vv_x}{c^2}\right)$

$$v_x' = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \gamma\left(\frac{dx}{dt} - v\right)\frac{1}{\gamma}\,\frac{1}{1 - \frac{vv_x}{c^2}} = \frac{v_x - v}{1 - \frac{vv_x}{c^2}}$$

So: $v_x' = \frac{v_x - v}{1 - \frac{vv_x}{c^2}}$ and $v_x = \frac{v_x' + v}{1 + \frac{vv_x'}{c^2}}$.

(Also v_y' and v_z' transforms because $dt' \neq dt$).

Examples: Show that if $v_x' = c$ then $v_x = c$!

Find v_x if $\frac{v_x'}{c} = \frac{v}{c} = 0.75$

What happens if $v_x' = -c$?

Interprete these examples with words of movement in relation to S and to S'.

What results would the Galileo Transformation give?

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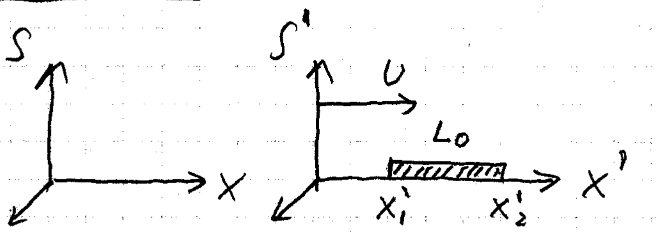
* Investigate the function $\gamma(v)$ or $\gamma(x) = \frac{1}{\sqrt{1 - x^2}}$, $\left(x = \frac{v}{c}\right)$ on your calculator!

$\left(\gamma > 1.\ 0 < \frac{1}{\gamma} \leq 1\right)$

The Lorentz Length Contraction

A stick is positioned at rest in S'. The coordinates in S' of the end-points are: x_1' and x_2'.



The length of the stick in S' (the "rest-length") becomes:

$$L_0 = x_2' - x_1' \quad (\text{"The proper length"})$$

To measure the length of the stick as seen from S we must find the coordinates of the end-points at one specific time t measured in S. The space-time events in S and S' are:

In S: (x_1, t) and (x_2, t) | $L = x_2 - x_1$

In S': (x_1', t_1') and (x_2', t_2') | (Same t in S!)

According to the Lorentz Transformation:

$$\left.\begin{aligned} x_1' &= \gamma(x_1 - ut) \\ x_2' &= \gamma(x_2 - ut) \end{aligned}\right\} \quad L_0 = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L$$

or: $\boxed{L = \frac{1}{\gamma} L_0 \quad (L \leq L_0)}$

The Lorentz Time Dilation

Two events occur at two different times t_1' and t_2', but in the same point of space x' in S'.

The time-interval between the two events becomes: $\tau_0 = t_2' - t_1'$ ("The proper time")

As seen from S the two events occur at the two different times t_1 and t_2, and at two different positions x_1 and x_2.

In S: (x_1, t_1) and (x_2, t_2) | $\tau = t_2 - t_1$

In S': (x', t_1') and (x', t_2') | (Same space-point in S'!)

According to Lorentz Transf. the time-interval in S becomes:

$$\left.\begin{aligned} t_1 &= \gamma\left(t_1' + \frac{u}{c^2}x'\right) \\ t_2 &= \gamma\left(t_2' + \frac{u}{c^2}x'\right) \end{aligned}\right\} \quad \tau = t_2 - t_1 = \gamma(t_2' - t_1')$$

or: $\boxed{\tau = \gamma \tau_0 \quad (\tau \geq \tau_0)}$

Remember: $\gamma = \dfrac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ $\qquad$ $\dfrac{1}{\gamma} = \sqrt{1 - \dfrac{u^2}{c^2}}$

$\gamma > 1$ $\qquad$ $0 < \frac{1}{\gamma} \leq 1$

Length Contraction – alternative

Let the observer in S measure how long time $t_2 - t_1$ it takes for the stick to pass a fixed point x in S. Then the length of the stick must be: $L = v(t_2 - t_1)$.

In S: (x, t_1) and (x, t_2) | $L = v(t_2 - t_1)$

In S': (x_2', t_1') and (x_1', t_2') | $L_0 = x_2' - x_1'$

$$\left.\begin{aligned} x_1' &= \gamma(x - v t_2) \\ x_2' &= \gamma(x - v t_1) \end{aligned}\right\} \quad L_0 = x_2' - x_1' = \gamma v (t_2 - t_1) = \gamma L$$

$$\text{or} \quad L = \frac{1}{\gamma} L_0$$

Time-Dilation – alternative

Let the observer in S measure the x-coordinates x_1 and x_2 of the two events occuring at x' in S' at times t_1' and t_2'. Then the time-interval as seen from S must be: $\tau = (x_2 - x_1)/v$.

In S: (x_1, t_1) and (x_2, t_2) | $\tau = \frac{x_2 - x_1}{v}$

In S': (x', t_1') and (x', t_2') | $\tau_0 = t_2' - t_1'$

$$\left.\begin{aligned} x_1 &= \gamma(x' + v t_1') \\ x_2 &= \gamma(x' + v t_2') \end{aligned}\right\} \quad \begin{aligned} x_2 - x_1 &= \gamma v (t_2' - t_1') \\ \tau &= \frac{x_2 - x_1}{v} = \gamma \tau_0 \end{aligned}$$

(See also 26-6)

The mu-meson (muon) μ

① Find the lifetime of the muon.

② Assuming the velocity of μ is 99.9% of c, find the distance it can travel during its lifetime

③ The μ is produced by cosmic radiation at the top of the atmospere, height 10km above Earth surface. Anyway a lot of them arrive down at the surface – in contradiction to ②. How can that be? – A paradox!

③ Let S' follow the μ, $v = 99.9\%$ of c. The lifetime from ① is the proper lifetime τ_0 of the μ.

Find the lifetime τ of the muon as seen from the Earth, S.

④ Find the distance the muon can travel during its lifetime τ as seen from S.

⑤ Explain the paradox in ③.

⑥ Explain what happens as seen from the muon!

Relativistic dynamics

Analyzing collisions of particles, assuming conservation of momentum, leads to the law of mass increase:

$$m = \gamma m_0 \quad \left(\gamma = \frac{1}{\sqrt{1-v^2/c^2}},\ m_0 : \text{restmass}\right).$$

Relativistic momentum defined as: $\vec{p} = m\vec{v}$.

Relativistic N2: $\vec{F} = \dfrac{d\vec{p}}{dt}$

(not here specifying the concept of force – it depends on the physics in focus. The force must also be transformed. E.g. in electro-magnetics: $F = qE + vB\sin\theta$ also in rel.).

More on N2, differentiating $\dfrac{d\vec{p}}{dt}$:

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \underbrace{\vec{v}\,\frac{dm}{dt}}.$$

Assuming E_{kin} still equal to net work of force it can be proved that:

$$E_{kin} = mc^2 - m_0c^2$$

Using the binomial expansion $\dfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x$ we find for small $\frac{v}{c}$: (the classical limit)

$$E_{kin} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}m_0c^2 - m_0c^2 \approx m_0c^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) - m_0c^2 = \frac{1}{2}m_0v^2$$

The total relativistic energy: $E = mc^2 = E_{kin} + m_0c^2$.

If m_0 consists of parts, m_0c^2 also includes potential energy – as you know from nuclear physics.

The energy – momentum formula:

$$\left.\begin{aligned} E &= mc^2 = \gamma m_0 c^2 \\ p &= mv = \gamma m_0 v \end{aligned}\right\} \quad E^2 = p^2c^2 + m_0^2c^4$$

Alternative units of mass and momentum:

Mass: MeV/c^2. Momentum: MeV/c

<u>Example</u> An electron accelerated through a potential difference of ~~[illegible]~~ $V = 2 \times 10^6$ V.

① Find the classical result of velocity.

② Use relativistic equation of kinetic energy $= Ve$.

Express γ by V, e, m_0, c using def. of E_{kin}

Calculate γ and v. (as a factor of c)

Trick: If you express energy in MeV and mass of electron in MeV/c^2, calculation of γ becomes very easy!

③ Find the final momentum of the electron.

a) Directly from results of ②.

b) Using $E^2 = p^2c^2 + m_0^2c^4$ and $E = E_{kin} + m_0c^2$

Trick: Try to express p in unit MeV/c.

To find v, first find m from $E = mc^2$,

then v from $p = mv$.

Doppler & Red Shift:

Sound (propagating in a medium) (Giancoli 3.ed, 12-8):

Moving source: (v_s velocity of source, v velocity of wave)

$\lambda' = \lambda\left(1 \mp \frac{v_s}{v}\right)$. $\quad f' = f\frac{1}{1 \mp \frac{v_s}{v}}$. (−: source towards observer; +: source away from observer).

Velocity of wave with respect to observer: $v' = \lambda' f' = \lambda f = v$.

Moving observer: (v_o velocity of observer)

$\lambda' = \lambda$. $\quad f' = f\left(1 \pm \frac{v_o}{v}\right)$ (+: ~~source~~ observer towards source; −: observer away from source)

$v' = \lambda' f' = \lambda f\left(1 \pm \frac{v_o}{v}\right)$.

Light (electromagnetic waves − no propagating medium)
(G 3.ed.: 33-4)

Source and observer moving relatively to each other.

(v: velocity of observer away from source)

$\lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$. This can be derived from the Lorentz equations. c independent!

If $\frac{v}{c} \ll 1$: $\lambda' \approx \lambda\left(1+\frac{v}{c}\right)^{\frac{1}{2}}\left(1+\frac{v}{c}\right)^{\frac{1}{2}} = \lambda\left(1+\frac{v}{c}\right)$ − classical limit.

If source and observer are moving (fast) away from each other, then $v > 0$, $\lambda' > \lambda$;* that is the color of visible light is shifted towards **red**, hence the term **redshift**.
(* or $f' < f$).

Gravitational red shift

− Simplified, see notes from IBID p. 676 − 678

Simplification: Using classical Doppler shift and assuming uniform gravitational field.

$$\frac{\Delta f}{f} = \frac{g\,\Delta h}{c^2}$$

(Δh change in height of grav. field
More gravitation ⇒ more redshift ⇒ lower f)

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Particles and photons

Particles, relativistic (and quantum mechanical)

$E = mc^2, \quad m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$

$p = mv$

$\left.\right\} E^2 = p^2c^2 + m_0^2c^4$

$\lambda = \dfrac{h}{p}, \quad \nu = \dfrac{E}{h} \; (E = h\nu). \quad \Big\} \; \lambda\nu = \dfrac{E}{p} = \dfrac{c^2}{v}.$

(not $\lambda\nu = v$!)

Photons (always relativistic, $v = c$)

$p = mv = \dfrac{E}{c^2}v = \dfrac{E}{c} \cdot \dfrac{v}{c}. \quad v \to c \Rightarrow p \to \dfrac{E}{c}.$

From $E^2 = p^2c^2 + m_0^2c^2$ with $E = pc$: $\quad m_0 = 0$!

$\lambda\nu = \dfrac{h}{p} \cdot \dfrac{E}{h} = \dfrac{E}{p} = c$

$m = \dfrac{E}{c^2} = \dfrac{h\nu}{c^2} = \dfrac{h}{\lambda c}.$

Particles, non-relativistic ($v \ll c$)

Binomial expansion of

$\dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \dfrac{1}{2}\dfrac{v^2}{c^2}$

$E = mc^2 \approx m_0c^2\left(1 + \dfrac{1}{2}\dfrac{v^2}{c^2}\right) = m_0c^2 + \dfrac{1}{2}m_0v^2$

$\lambda = \dfrac{h}{p}.$ (Also $\nu = \dfrac{E}{h}$, but E includes m_0c^2!)

$p = mv \approx m_0v\left(1 + \dfrac{1}{2}\dfrac{v^2}{c^2}\right) \approx m_0v$